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Technical note: Solving inventory models by algebraic method

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ABSTRACT

We consider the open question raised by Chang et al. (2005) to solve the EOQ and EPQ inventory models without referring to calculus. Lau et al. (2016) and Chiu et al. (2017) both extended this open question by deriving criterion for the existence and uniqueness of the interior minimum solution but they used analytical techniques that are related to calculus. Moreover, their derivations are incomplete and contained questionable results. In this note, we only used algebraic approach for their extended open question.

1. Introduction

Since Grubbström and Erdem (1999) used an algebraic approach to solve the minimum problem of an inventory model, there are nearly two hundred papers that followed this trend to develop different algebraic methods to find the optimal solutions for inventory systems. Most papers concentrated on their own inventory models and did not pay attention to an open question proposed by Chang et al. (2005), which contains a quadratic polynomial inside a square root and requires finding the optimal solution only by algebraic methods without referring to calculus. Until recently, Lau et al. (2016) reconsidered the open question and extended to a more general setting. They tried to obtain criterion to guarantee the existence and uniqueness of the optimal solution for the minimum problem. Chiu et al. (2017) pointed out the findings of Lau et al. (2016) contained questionable results with improper analytic approach and inadequate partitioning of the solution space, and provided an improvement. However, we find that the improvement of Chiu et al. (2017) is right but incomplete. Moreover, they used calculus method to obtain their results. In Chang et al. (2005), the original restriction is to derive the optimal solution by algebraic approach without referring to calculus.

The purpose of this note is to present a further revision of Chiu et al. (2017) with algebraic approach and answer the open question proposed by Chang et al. (2005) as a corollary. A major contribution of this technical note is our proposed derivation method that will be useful for researchers and practitioners to solve the minimum problems under algebraic approaches. In Section 2, a brief review of merit achievements associated with the inventory modeling for the past decade is presented. It is followed by our proposed methodology in Section 3, and numerical applications in Section 4. Finally, a synthesis discussion in Section 5 will conclude our technical note.

2. Brief review for inventory modeling achievements

To be compatible with Càrdenas-Barrón (2001), Chang et al. (2005), Lau et al. (2016) and Chiu et al. (2017), we use the same notation and expressions as theirs. Readers for this note are suggested referring to them for notation and assumptions.

The original inventory model solved by algebraic method was proposed by Càrdenas-Barrón (2001), and further revised by Ronald et al. (2004) and Chang et al. (2005) as

$$C(Q,B) = \frac{b+h}{2\rho Q}B^2 - hB + \frac{h\rho}{2}Q + \frac{KD}{Q} + cD. \tag{1}$$

Specifically, Chang et al. (2005) mentioned that an alternative way to solve their minimum problem is to solve the following problem:

$$C(Q(B),B) = cD + h\left(\sqrt{\left(1+\left(\frac{b}{h}\right)\right)B^2 + \frac{2\rho}{h}KD} - B\right). \tag{2}$$

By Equation (2), they provided the next open minimum problem:

$$\sqrt{(1+\alpha)B^2 + \beta} - B \tag{3}$$

without using partial derivatives of calculus, where $\alpha = b/h$, $\beta = 2\rho KD/h$, $\alpha > 0$ and $\beta > 0$.

Recently, there are two papers: Lau et al. (2016) and Chiu et al. (2017) to consider the following more generalized minimum problem:

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$$f(x) = \sqrt{ax^2 + bx + c} - x \tag{4}$$

with $f(x) > 0$, for $x > 0$, to secure the minimum problem has an interior optimal solution.

Lau et al. (2016) obtained two cases:

(a) $a > 1$, $b \le 0$, $c > 0$ and $4ac > b^2$, and
(b) $a > 1$, $b > 0$, $c > 0$ and $4c > b^2$.

However, their solution contained questionable results, which will be demonstrated by our Theorem 1 in Section 3, and several of their derivations were derived by calculus.

Chiu et al. (2017) claimed two cases:

(a) $a > 1$, $c > 0$ and $4ac > b^2$, and
(b) $a > 1$ and $4c > b^2$.

They used the knowledge of calculus in derivations and their findings are incomplete, which will be demonstrated by our Theorem 1 as well. Hence, in this note, we will provide further improvements for Lau et al. (2016) and Chiu et al. (2017) with algebraic method.

3. Proposed algebraic method

Our goal is to find conditions to guarantee that $f(x) = \sqrt{ax^2 + bx + c} - x$ for $x > 0$ with $f(x) > 0$ has a unique minimum (optimal) solution by algebraic methods. Before proving our theorem, we would firstly explain the reason why we concentrate on $x > 0$ with $f(x) > 0$ herein.

When $x \to 0^+$ and $f(x) \to 0$, it implies that the inferior value occurs on the boundary. Thus, the original inventory model has an inferior value when $Q \to 0^+$, which is violating the common sense of the original minimum cost inventory model. As a Remark, for the original minimum cost inventory model, the average set up cost will go to infinite when the replenishment cycle approaches to zero. Therefore, the original minimum cost inventory model cannot have the finding as $Q^* \to 0^+$. If the optimal solution of $f(x)$ satisfies $x^* \to \infty$, it implies that the optimal order quantity as well as the holding cost will go to infinite. Thus, $x^* \to \infty$ is not an acceptable optimal solution for $f(x)$. Under the condition of a point, say $\hat{x}$, satisfying $0 < \hat{x} < \infty$ and $f(\hat{x}) \le 0$, the corresponding inventory model will have negative or zero holding cost and shortage cost that is a contradiction for inventory models. Hence, we look for restrictions to guarantee $x > 0$ and $f(x) > 0$.

Remark. We want to rule out the case of $a = 1$, $c \ge 0$ and $b = 2\sqrt{c}$ to result in the trivial case $f(x) = \sqrt{x^2 + 2\sqrt{c}x + c} - x \equiv \sqrt{c}$ that has infinite minimum solutions.

In this section, we will prove two necessary conditions of $c > 0$ and $a > 1$, and rewrite Equation (4) to obtain additional three conditions: $4ac - b^2 > 0$, $4c > b^2$ when $b \ge 0$, and $4(a-1)c > b^2$ when $b < 0$ for deriving our proposed Theorem 1.

First, we will prove that $c > 0$. Assuming $c < 0$, we select a point x_1 with $x_1 > 0$ and $ax_1^2 + bx_1 + c < 0$, which is a contradiction with $\sqrt{ax_1^2 + bx_1 + c}$.

Thus, we select $x_1 = \frac{1+|b|}{1+|a|} s$ with $s = \min\left\{1, \frac{|c|(1+|a|)}{2(1+|b|)^2}\right\}$. As $s \le 1$, we compute

$$\begin{aligned} ax_1^2 + bx_1 + c &< (1+|a|)x_1^2 + (1+|b|)x_1 - |c| = \frac{(1+|b|)^2}{1+|a|}(s^2+s) - |c| \\ &\le 2s\frac{(1+|b|)^2}{1+|a|} - |c| \le 0 \end{aligned} \tag{5}$$

to imply an unacceptable result $f(x_1) < 0$.

With an assumption of $c = 0$, we construct a sequence (t_n) as $t_n = \frac{1}{n^2}$ and $t_n \le \sqrt{t_n}$, and evaluate that

$$\begin{aligned} f(t_n) &= \sqrt{at_n^2 + bt_n} - t_n \le \sqrt{|a|t_n^2} + \sqrt{|b|t_n} + t_n \le (1+|a|+|b|)\sqrt{t_n} \\ &= \frac{1+|a|+|b|}{n} \end{aligned} \tag{6}$$

to yield $f(t_n) \to 0$. As it will result in a solution of $x^* \to 0^+$, we derive the first condition of

$$c > 0. \tag{7}$$

Second, we will prove that $a > 1$. When $a < 1$, we take $x_0 = m\left(\frac{1+|b|}{1-a}\right)$ with $m = 1 + \frac{\sqrt{(1-a)|c|}}{1+|b|} > 1$ to imply that

$$\begin{aligned} (1-a)x_0^2 - (1+|b|)x_0 &= (1-a)m^2\left(\frac{1+|b|}{1-a}\right)^2 - (1+|b|)m\left(\frac{1+|b|}{1-a}\right) \\ &= \frac{(1+|b|)^2}{1-a}m(m-1) > \frac{(1+|b|)^2}{1-a}(m-1)^2 \\ &> \frac{(1+|b|)^2}{1-a}\frac{(1-a)|c|}{(1+|b|)^2} = |c| \ge c. \end{aligned} \tag{8}$$

From $(1-a)x_0^2 > (1+|b|)x_0 + c > bx_0 + c$, it yields $f(x_0) < 0$ that is a contradiction, and thus, $a < 1$ is not acceptable. The condition of $a \ge 1$ is derived. We further show that when $a = 1$, a minimum solution for $0 < x < \infty$ cannot be found. Note that the inferior value occurs on the boundary $x = 0$ or $x = \infty$ are out of the domain $0 < x < \infty$. When $a = 1$, three cases are elaborated separately herein: (a) $b > 2\sqrt{c}$, (b) $b < 2\sqrt{c}$ and (c) $b = 2\sqrt{c}$.

For case (a), when $a = 1$ and $b > 2\sqrt{c}$, we derive that

$$\begin{aligned} &4c < b^2 \Leftrightarrow ax^2 + bx + c > x^2 + 2\sqrt{c}x + c \Leftrightarrow \sqrt{ax^2 + bx + c} > x + \sqrt{c} \Leftrightarrow f(x) \\ &> \sqrt{c}. \end{aligned} \tag{9}$$

Thus, the inferior value occurs when $x \to 0^+$.

For case (b), $a = 1$ and $b < 2\sqrt{c}$, we show that $f(x) > b/2$ for all $x > 0$ in

$$\begin{aligned} f(x) > b/2 &\Leftrightarrow \frac{bx+c}{\sqrt{x^2+bx+c}+x} > \frac{b}{2} \Leftrightarrow bx + c \\ &> \frac{b}{2}\sqrt{x^2+bx+c} + \frac{b}{2}x \Leftrightarrow \frac{b}{2}x + c > \frac{b}{2}\sqrt{x^2+bx+c} \Leftrightarrow bcx + c^2 \\ &> \frac{b^2}{4}(bx+c) \Leftrightarrow c(bx+c) > \frac{b^2}{4}(bx+c) \Leftrightarrow (bx+c)\left(c - \frac{b^2}{4}\right) > 0. \end{aligned} \tag{10}$$

Next, we consider $f(x)$ when $x \to \infty$. Since $f(x) = \frac{b+(c/x)}{\sqrt{1+(b/x)+(c/x^2)}+1}$, $f(x) \to \frac{b}{2}$ when $x \to \infty$. Thus, the inferior value happens when $x \to \infty$.

For case (c), if $a = 1$ and $b = 2\sqrt{c}$, $f(x)$ is a constant function with $f(x) \equiv \sqrt{c}$. Thus, every positive point is considered as the optimal solution, which violates our goal of finding a unique minimum solution. Consequently, we derive the second condition

$$a > 1. \tag{11}$$

We rewrite Equation (4) as

$$\sqrt{ax^2 + bx + c} = x + f(x) \tag{12}$$

and take square on both sides. We arrange the expression in the descending order of x and treat $f(x)$ as a constant term for the moment. Through the square for x, it implies

$$4(a-1)^2\left(x-\frac{2f(x)-b}{2(a-1)}\right)^2+4c(a-1)-b^2+4bf(x)=4af^2(x). \tag{13}$$

We complete the square for $f(x)$ to yield that

$$4(a-1)^2\left(x-\frac{2f(x)-b}{2(a-1)}\right)^2+\left(\frac{a-1}{a}\right)(4ac-b^2)=4a\left(f(x)-\frac{b}{2a}\right)^2. \tag{14}$$

Based on Equation (14), owing to the coefficients of two square terms, $4(a-1)^2$ and $4a$ are both positive, we find a condition to obtain a positive minimum value as

$$4ac-b^2>0. \tag{15}$$

We derive the relation between the optimal point, x^* and the optimal value $f(x^*)$ as

$$x^*=\frac{2f(x^*)-b}{2(a-1)}. \tag{16}$$

With the restriction of the domain for $x^*>0$, we derive that

$$f(x^*)>b/2. \tag{17}$$

Therefore, we recall Equation (14) with its relation to Equations (16) and (17), and find that

$$f(x^*)-\frac{b}{2a}=\frac{-1}{2a}\sqrt{(a-1)(4ac-b^2)} \tag{18}$$

is not an acceptable value. Hence, we find that

$$f(x^*)-\frac{b}{2a}=\frac{1}{2a}\sqrt{(a-1)(4ac-b^2)}. \tag{19}$$

Consequently, referred to Equations (16) and (19), we obtain

$$x^*=\frac{1}{2a}\left(\sqrt{\frac{4ac-b^2}{a-1}}-b\right). \tag{20}$$

Based on Conditions of Equations (11) and (15), we know that $\sqrt{(4ac-b^2)/(a-1)}$ is well defined. For the well defined solution x^*, we need to ensure that $x^*>0$. Thus, we derive the following condition.

Lemma 1. *$x^*=\frac{1}{2a}\left(\sqrt{\frac{4ac-b^2}{a-1}}-b\right)$ is well defined if and only if (a) $b<0$, or (b) $b\geq 0$ with $4c>b^2$.*

Proof. We need to check $x^*>0$. When $b\geq 0$, we want $\frac{4ac-b^2}{a-1}>b^2$ that is $4c>b^2$, owing to $a>1$. When $b<0$, no extra requirement is needed.

We examine the derivations of Chiu et al. (2017) to know that they did not evaluate $f(x^*)$. Hence, we first find that

$$a(x^*)^2+bx^*+c=\frac{4ac-b^2}{4(a-1)}. \tag{21}$$

And then we compute

$$f(x^*)=\frac{b+\sqrt{(4ac-b^2)(a-1)}}{2a}. \tag{22}$$

To derive a reasonable positive minimum value, we have to check the minimum value of Equation (22), which is positive. Two cases are elaborated herein: $b\geq 0$ and $b<0$. When $b\geq 0$, we know that $f(x^*)>0$ without any additional requirement under the restriction of our Lemma 1 where $a>1$ and $4c>b^2$.

When $b<0$, we have to ensure the positivity of Equation (22) and then we simplify $(4ac-b^2)(a-1)>b^2$ as

$$4(a-1)c>b^2. \tag{23}$$

We combine our results in the next theorem.

Theorem 1. *For the existence and uniqueness of an interior minimum, we obtain the necessary conditions as $a>1$ and $c>0$.*

(i) *When $b\geq 0$, we find that $4c>b^2$, and*
(ii) *When $b<0$, we derive that $4(a-1)c>b^2$,*

with the minimum point, x^* of Equation (20) and minimum value, $f(x^*)$ of Equation (22).

Proof. When $b\geq 0$, the condition of $4ac>b^2$ is unnecessary. Because we already have $a>1$ and $4c>b^2$ to imply that $4ac>4c>b^2$. When $b<0$, the condition of $4ac>b^2$ is already contained in the inequality of $4(a-1)c>b^2$ since $4ac>4(a-1)c$.

Next, we come back to the open question proposed by Chang et al. (2005). We assume that $a=1+\alpha$, with $\alpha>0$, $b=0$ and $c=\beta>0$, to know that the conditions $a>1$, $4c-b^2=4\beta>0$ and $4(a-1)c-b^2=4\alpha\beta>0$ are all satisfied. We finally derive

$$f(x^*)=\sqrt{\frac{\alpha\beta}{1+\alpha}} \tag{24}$$

and

$$x^*=\sqrt{\frac{\beta}{\alpha(1+\alpha)}}. \tag{25}$$

It concludes that our proposed method without referring to calculus can derive the same results as predicted by Chang et al. (2005).

4. Numerical examples and an application of our derivation

In this section, we will provide four examples, the first two examples illustrate that the conditions of Lau et al. (2016) and Chiu et al. (2017) are questionable. The third example is used to demonstrate our findings for $b\geq 0$ while the fourth example is set for $b<0$.

For the first numerical example, we assume $a=2$, $b=2$ and $c=1$ so that the conditions of $a>1$ and $4ac>b^2$ proposed by Lau et al. (2016) and Chiu et al. (2017) are satisfied. However, we find that minimum of $f(x)=\sqrt{2x^2+2x+1}-x$ is with inferior value $f(0)=1$. Because for $x>0$, $\sqrt{2x^2+2x+1}>x+1$ if and only if $x^2>0$, there is no interior minimum point. Hence, we demonstrate that the criterion proposed by Lau et al. (2016) and Chiu et al. (2017) contained questionable results.

Next, for the second numerical example, we assume $a=2$, $b=-4$ and $c=3$ so that the conditions of $a>1$ and $4ac>b^2$ proposed by Lau et al. (2016) and Chiu et al. (2017) are satisfied. However, we find the minimum of $f(x)=\sqrt{2x^2-4x+3}-x$ is at the minimum point $x^*=1+(\sqrt{2}/2)$ and minimum value $f(x^*)=-1-(\sqrt{2}/2)$. It is against the purpose of extension by Lau et al. (2016) and Chiu et al. (2017) for deriving an interior minimum point with positive minimum value. Although the minimum point $x^*=1+(\sqrt{2}/2)$ is acceptable, the minimum value $f(x^*)=-1-(\sqrt{2}/2)$ is out of the scope of their original purpose.

For the third numerical example, we assume $a=2$, $b=2$ and $c=2$ so that our proposed criteria of $a>1$, $c>0$, $b\geq 0$ and $4c>b^2$ in Theorem 1 Part (i) are satisfied. We find $x^*=(\sqrt{3}-1)/2$ by Equation (20), and obtain $f(x^*)=(\sqrt{3}+1)/2$ through Equation (22).

For the fourth numerical example, we assume $a=2$, $b=-4$ and $c=5$ so that our proposed criteria of $a>1$, $c>0$, $b<0$ and $4(a-1)c>b^2$ in Theorem 1 Part (ii) are satisfied. We find $x^*=(\sqrt{6}+2)/2$ by Equation (20), and obtain $f(x^*)=(\sqrt{6}-2)/2$ through Equation (22).

We recall the inventory model of Cárdenas-Barrón (2010) and Tuan and Himalaya (2016),

$$TC(Q,B)=\frac{Ad}{Q}+\frac{h(Q-B)^2}{2Q}+\frac{vB^2}{2Q} \tag{26}$$

and compare the expression of Equation (26) with Equation (1). We find that if we assume $\rho=1$, $b=v$, $C=0$, $D=d$ and $K=A$, we can convert the inventory model proposed by Cárdenas-Barrón (2001) to that of Cárdenas-Barrón (2010) and Tuan and Himalaya (2016). Hence, our derivation can be applied to Cárdenas-Barrón (2010) and Tuan and Himalaya (2016) to provide an alternative approach for solving inventory models by algebraic methods.

5. Conclusion

In this note, we solved the generalized open question proposed by Lau et al. (2016) also point out the error questionable results of Lau et al. (2016) and Chiu et al. (2017). Consequently, the original open question of Chang et al. (2005) becomes a special case from our solution. Our derivation of Equation (20) for the optimal solution of x^* by algebraic method is the same result as predicted by Chang et al. (2005) by different algebraic method and that of Chiu et al. (2017) by analytical approach. Hence, we provide a reply for the open question proposed by Chang et al. (2005) and a generalized open question proposed by Lau et al. (2016). Moreover, we revise criteria discussed by Lau et al. (2016) and Chiu et al. (2017) to guarantee the existence and uniqueness of the interior optimal solution.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this note.

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